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THE STABILITY OF COLLINEAR LIBRATION POINTS IN THE PHOTOGRAVITATIONAL THREE-BODY PROBLEM†

A. L. KUNITSYN

Moscow

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The stability of the relative equilibrium positions (collinear libration points) of the restricted circular photogravitational three-body problem, in which a passively gravitating point, in addition to experiencing the Newtonian gravity force from the main bodies (stars) also experiences forces of light pressure from each of them [1], is investigated. Previously obtained [2] conditions of stability are analysed from new viewpoints, enabling them to be presented more clearly. This is achieved by introducing, for each fixed pair of main bodies, a certain generalized parameter (which was used earlier [3, 4] in an examination of triangular libration points) and by transferring from the parameter space of the system to its configuraion space.

The question of the existence and stability of libration points – the relative equilibrium positions in a system of coordinates rotating together with the main bodies – of the restricted photogravitational three-body problem has been in a number of papers (the most complete of these is given in [1]). As a result of comparatively recent investigations, carried out by various researchers, it has been established that if, within the framework of the model of the classical restricted three-body problem, the additional potential force field of light repulsion is taken into account, an entire family of new libration points is obtained, the coordinates of which are determined both by the parameters of the gravitational–repulsive field of the main bodies and by the "sail capacity" of the particles located in this field, represented by the ratio of the characteristic area of a particle to its mass. Part of this family is a set of libration points similar to two triangular libration points of the classical restricted three-body problem, while the other part is a set of points positioned on a straight line passing through the main bodies (regarded as point masses) and similar to five collinear libration points of the classical problem. It was shown in [2] that not only triangular libration points can be stable but also collinear libration points, which in the classical problem are always unstable. An extremely simple geometric interpretation of the conditions of stability of triangular libration points was given in [4], including the elliptical case of the problem, where the eccentricity of the orbit of the main bodies is small. This proved to be possible by transferring from the parameter space of the system to its configuration space, and by introducing a generalized parameter that characterized (along with the mass parameter used in the classical problem, which comprises the relative mass of one of the main bodies) the power of the radiation of each fixed pair of main bodies. These considerations make it possible to simplify considerably, and make physically clearer, the analysis of the stability and of the collinear libration points of the restricted circular photogravitational three-body problem.

As is well known [1], the gravitational–repulsive force field of the restricted circular photogravitational three-body problem can be specified by the force function

$$W = (x^2 + y^2)/2 + Q_1(1-\mu)/R_1 + Q_2\mu/R_2$$
$$Q_i = (F_i^g - F_i^p)/F_i^g, \quad R_i = (x-a_i)^2 + y^2 + z^2, \quad i = 1,2 \tag{1}$$

where x, y an z are the dimensionless (the distance between the main bodies is taken as the unit of length) Cartesian baricentric coordinates of a passively gravitating particle P in an $Oxyz$ reserence system rotating uniformly about the Z axis with an angular velocity equal to unity, μ and $1-\mu$ are the relative masses of the main bodies S_2 and S_1 (regarded as point masses), related to their total mass, $a_1 = -\mu$ and $a_2 = 1-\mu$ are their dimensionless coordinates and Q_i are the reduction coefficients of the mass of the

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particle P, characterizing the influence of the repulsive field of the light pressure, where F_i^p and F_i^g are the forces of gravity and light pressure respectively. These coefficients can also be represented in the form

$$Q_i = 1 - q_i\sigma, \quad q_i = C_i/(fM_i), \quad \sigma = (1+\varepsilon)s/m \tag{2}$$

where f is the gravitational constant, M_i is the mass of the body S_i, C_i is a constant coefficient characterizing the power of its radiation and σ is the sail capacity of a particle P (m and s are respectively its mass and characteristic cross-sectional area, and ε is the reflection coefficient).

As can be seen, the influence of light pressure increases as the absolute particle size decreases and can be as large as desired even for particles of high density. It is also obvious that the physically possible value of the reduction factor cannot exceed unity ($Q_i = 1$ if there is no light pressure from the body S_i).

From relations (2) it follows that, for any fixed pair of bodies S_1 and S_2, the reduction factors of particle P with any sail capacity must satisfy the relation

$$(1-Q_2)/(1-Q_1) = q_2/q_1 = k$$

From this it follows that, for any fixed pair of bodies S_1 and S_2, the reduction factors cannot take arbitrary values, a fact to which attention was drawn in [3, 4] when investigating the stability of triangular libration points. A consideration of arbitrary, unrelated values of Q_1 and Q_2 (as was done in all previous papers) involves the simultaneous examination not only of arbitrary points but also of various bodies S_1 and S_2, which hinders a clear physical interpretation of the results obtained. The parameter k, equal to the ratio of the "specific" radiation powers of the main bodies, must be regarded as an additional characteristic (to the mass parameter μ) of the gravitational–repulsive field for a–fixed pair of main bodies. Obviously, k can take any non-negative values. When $k = 0$, we will have the case of a single radiating body. When $k = 1$, the "specific" radiation powers of both bodies are the same, from which it follows that $Q_1 = Q_2$. In practice, this is the most important case and will be examined in this paper.

The equilibrium conditions, from which it is possible to determine the coordinates of collinear libration points, are found from the requirement that the first variation of the force function W must vanish, which, when $y = z = 0$, leads to the equation (summation over i is carried out from 1 to 2)

$$x - \sum a_i Q_i (x - a_i)/R_i^3 = 0 \tag{3}$$

A detailed analysis of Eq. (3) showed [2, 3] that, depending on the values of the reduction factors, both internal collinear libration points (positioned between the bodies S_1 and S_2) and external ones (positioned outside the segment S_1S_2) are possible.

The stability conditions for the collinear points (derived from first-approximation equations) for all possible values of Q_1 and Q_2 are given by the inequalities [2]

$$8/9 \leqslant A \leqslant 1, \quad A = \Sigma a_i Q_i / R_i^3 \tag{4}$$

As in the classical problem, these conditions do not give secular stability and are only the conditions for gyroscopic stabilization. However, since the system in question belongs to the Hamiltonian class, these conditions mean complete Birkhoff stability, i.e. such a stability that holds when non-linear terms of as high (finite) an order in the equations of perturbed motion, as desired are retained with the exception, possibly, of the set of values of the parameters corresponding to second- and third-order resonance. Note, however, that non-satisfaction of conditions (4) means strict Lyapunov instability.

We will show that the external libration points are unstable for any values of Q_1 and Q_2. Consider the libration points with coordinates $x < a_1 = -\mu$. Taking into account also that $x = a_1 - R_1$, we substitute the value of Q_1 determined by equilibrium condition (3), into the expression for A. This gives

$$A = 1 + \mu(1 - Q_2/R_2^3)/R_1$$

The condition of stability $A \leqslant 1$ leads to the inequality $Q_2 \geqslant R_2^3$. However, since in the case considered $R_2 = 1 + R_1$, we finally obtain $Q_2 > 1$, which falls outside the range of physically permissible values of Q_2. Since this conclusion holds for any values of μ, external libration points positioned to the right of the body S_2 will also be unstable.

We will now consider internal points whose coordinates satisfy the inequalities $-\mu < x < 1 - \mu$. We will first examine the completely symmetrical case when $Q_1 = Q_2 = Q$ and $\mu = 1/2$. The equilibrium condition (3) is now written in the form

$$2x + Q(R_1^{-3} - R_2^{-3}) = 0 \tag{5}$$

from which it can be seen that, for the values of x cansidered, apart from $x = 0$, we will always have $Q < 0$, and consequently

$$2A = Q(R_1^{-3} + R_2^{-3}) < 0$$

which does not satisfy the condition of stability (4). Thus, in the case considered, only the origin of coordinates ($R_1 = R_2$) can be a stable libration point.

From Eq. (5) it follows that at this libration point there may be particles with any value of Q; here, only some of them may be stable. In fact, when $R_1 = R_2 = 1/2$, we will have $A = 8Q$, and stability condition (4) gives

$$1/9 \leqslant Q \leqslant 1/8 \tag{6}$$

i.e. in the completely symmetrical case there may in fact be an innumerable set of particles with different sail capacity at the origin of coordinates, the reduction factor of which satisfies stability condition (6). It is interesting to note that $Q = 1/8$ is the limiting minimum value for all triangular libration points possible in this case (as shown earlier [4], all triangular libration points in this case are situated on the y axis; their coordinates satisfy the inequalities $-\sqrt{3/2} < y < \sqrt{3/2}$, and Q takes all values from $1/8$ to 1 when $y = \pm\sqrt{3/2}$). In this case, only those points for which the inequality [4]

$$36\mu(1-\mu)\sin^2(\psi_1 + \psi_2) \leqslant 1$$

is satisfied will be stable. Here ψ_1 and ψ_2 are the angles the vectors R_1 and R_2 make with the x axis. In the case in question ($\mu = 1/2$, $\psi_1 = \psi_2 = \psi$), this condition gives

$$\sin^2\psi \leqslant \sin^2\psi^* = 1/2 - \sqrt{2/3}$$

i.e. all triangular libration points whose ordinates satisfly the condition $|y| \leqslant y^* = 0.085786\ldots$ will be stable. Since for triangular libration points the relations $R^3 = Q$[4] are satisfied, in the case considered the reduction factor of particles situated at these points decreases from the maximum value, equal to $1/(8\cos^3\psi^*)$ and corresponding to $y = y^*$, to $1/8$ at the origin of coordinates, where, furthermore, there may be an innumerable set of particles whose reduction factor satisfies inequalities (6).

We will now consider the case of arbitrary values of μ ($0 < \mu < 1/2$), retaining the condition $k = 1$ ($Q_1 = Q_2 = Q$). Equilibrium condition (3) can then be written in the form

$$Q[\mu/R_1^2 - (1-\mu)/R_2^2] + x = 0 \tag{7}$$

whence, when $R_1 = R_2$ ($x = 1/2 - \mu$), we obtain $Q = 1/8$, irrespective of the value of μ. According to inequality (6), the value of Q obtained (as in the case when $\mu = 1/2$) lies at the boundary of the stability region. However, unlike the case when $\mu = 1/2$, only one particle with a sail capacity corresponding to the Q value found can now be situated at this libration point.

Equation (7) is also satisfied by the values $x = Q = 0$ and, consequently, according to inequality (6), the origin of coordinates is now always unstable. We will show that all points for which $x = 0$ are also unstable. Since for stability it is necessary that $Q > 0$, Eq. (7) implies the inequality

$$\mu/R_1^2 - (1-\mu)/R_2^2 > 0 \tag{8}$$

from which it follows that $\mu > (1 + \rho^2)^{-1}$ should hold, where $\rho = R_1/R_2$. The latter inequality with $x < 0$ ($\rho < 1$) implies that $\mu > 1/2$. However, the latter inequality contradicts the range of variation of μ considered, and thus the stability conditions in the range of variation of x and μ considered cannot be satisfied.

All libration points with $x > 0$, for which $\rho > 1$, will be unstable. In fact, for positive x and Q, as follows from Eq. (7), inequality (8) is reversed and can be satisfied for all $\mu < 1/2$. Substituting the value of Q given by Eq. (7) into the expression for A, taking the above into account, from the stability condition $A < 1$ we will have

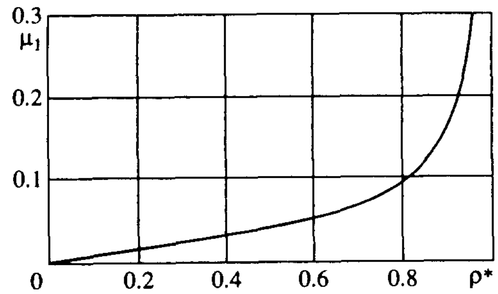

Fig. 1

$$[(1-\mu)/R_1^3+\mu/R_2^3]x\leqslant(1-\mu)/R_1^2-\mu/R_2^2 \tag{9}$$

from which it follows that $\rho < 1$.

Hence, it remains to consider the range $\mu/(1-\mu) < \rho < 1$, where the opposite inequality to (8) is satisfied and where, according to inequality (9), $A \leqslant 1$. Substituting the value of Q from Eq. (7) into stability condition (6) $A > 8/9$, we will have

$$[(1-\mu)/R_1^3+\mu/R_2^3]x>8[(1-\mu)/R_1^2-\mu/R_2^2]/9$$

The inequality obtained can be reduced to the form

$$f(\mu,\rho)=\mu^2-[1+b(1+\rho^2)]\mu+b\geqslant 0 \tag{10}$$

$$b=\rho(1+\rho)^{-1}(1-\rho^3)/9$$

Since in the range considered ($\rho < 1$) we have $b > 0$, the polynomial $f(\mu, \rho)$ always has, in relation to μ, two positive roots $\mu_1(\rho)$ and $\mu_2(\rho) > \mu_1$, and inequality (10) is satisfied if $0 < \mu \leqslant \mu_1(\rho)$ or $\mu \geqslant \mu_2(\rho)$. It can be shown that, for all the ρ values considered, the larger root, which emanates from the range of variation of μ, considered, is greater than unity, and consequently, for stability, only the one inequality $\mu \leqslant \mu_1(\rho)$ must be satisfied.

Figure 1 shows a graph of $\mu_1(\rho)$, which is the boundary of the stability region, situated below this curve. The above analysis enables us to conclude that, for any fixed $\mu < 1/2$, stable collinear libration points can only exist in the range $[\rho^*, 1]$ (contracting to a point as $\mu \to 1/2$), where

$$\rho^*>\mu_1/(1-\mu_1)$$

and μ_1 is the smaller root of polynomial (9) when $\rho = \rho^*$. The reduction factors of particles at points in this range can be found using Eq. (7).

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